

ON RATIONALLY ERGODIC AND RATIONALLY WEAKLY MIXING RANK-ONE TRANSFORMATIONS

IRVING DAI, XAVIER GARCIA, TUDOR PĂDURARIU, AND CESAR E. SILVA

ABSTRACT. We study the notions of weak rational ergodicity and rational weak mixing as defined by Jon Aaronson. We prove that various families of infinite measure-preserving rank-one transformations possess (or do not possess) these properties, and consider their relation to other notions of mixing in infinite measure.

1. DEFINITIONS AND PRELIMINARIES

Let (X, \mathcal{B}, m) be a standard Borel measure space with a σ -finite nonatomic measure m . In most cases, we will assume that m is infinite. A transformation $T : X \rightarrow X$ is **measurable** if $T^{-1}A \in \mathcal{B}$ for all $A \in \mathcal{B}$. A measurable transformation T is **measure-preserving** if $m(A) = m(T^{-1}A)$ for all $A \in \mathcal{B}$. We say that T is **ergodic** if every T -invariant set (i.e., $T^{-1}A = A \bmod m$) is null ($m(A) = 0$) or full ($m(X \setminus A) = 0$). We say that T is **conservative** if for every measurable set A of positive measure, there exists a positive integer n such that $m(A \cap T^{-n}A) > 0$. It follows that T is conservative and ergodic if and only if for every set A of positive measure, $\bigcup_{n=0}^{\infty} T^{-n}A = X \bmod m$. An **invertible measurable transformation** is a measurable transformation whose inverse is also measurable. Throughout this paper, we will assume that T is an invertible, conservative ergodic, measure-preserving transformation on (X, \mathcal{B}, m) , and we will typically use the forward images $T^n A$ instead of $T^{-n}A$.

When T is a measure-preserving transformation on a probability space X , the Birkhoff ergodic theorem states that ergodicity is equivalent to having the convergence

$$(1) \quad \frac{1}{n} \sum_{k=0}^{n-1} m(A \cap T^k B) \rightarrow m(A)m(B)$$

for all measurable $A, B \subset X$. This gives a quantitative estimate for the average number of visits of one set to another. When X has infinite measure, however, the Birkhoff ergodic theorem implies that the Cesaro averages of

2000 *Mathematics Subject Classification.* Primary 37A40; Secondary 37A05.

Key words and phrases. Infinite measure-preserving, ergodic, rationally ergodic, rank-one.

(1) converge to 0 for all pairs A, B of finite measure. Moreover, in [1] Aaronson proved that there exists no sequence of normalizing constants for which the averages of (1) converge to $m(A)m(B)$, and he proposed in turn the definitions of rational ergodicity and weak rational ergodicity.

For any measurable set $F \subset X$ of finite positive measure, define the **intrinsic weight sequence** of F to be

$$u_n(F) = \frac{m(F \cap T^n F)}{m(F)^2}$$

and write

$$a_n(F) = \sum_{k=0}^{n-1} u_k(F).$$

A transformation T is said to be **weakly rationally ergodic** (see [1]) if there exists a measurable set $F \subset X$ of positive finite measure such that for all measurable $A, B \subset F$, we have

$$(2) \quad \frac{1}{a_n(F)} \sum_{k=0}^{n-1} m(A \cap T^k B) \rightarrow m(A)m(B)$$

as $n \rightarrow \infty$. If this convergence happens only along a subsequence $\{n_i\}$ of \mathbb{N} , we say that T is **subsequence weakly rationally ergodic**. To emphasize the set F , we will sometimes say T is **weakly rationally ergodic on F** . Note that any measure-preserving ergodic transformation on a probability space is trivially weakly rationally ergodic, by taking F to be the whole space itself. Then $a_n(F) = n$, so (2) reduces to the Cesaro sum definition of ergodicity.

A transformation T is said to be **(spectrally) weakly mixing** if whenever $f \in L^\infty(X, m)$ and $f \circ T = zf$ for some $z \in \mathbb{C}$, then f is constant a.e. When X is a probability space, this is equivalent to ergodicity of the Cartesian square and also to the strong Cesaro convergence

$$\frac{1}{n} \sum_{k=0}^{n-1} |m(A \cap T^k B) - m(A)m(B)| \rightarrow 0$$

for all measurable $A, B \subset X$. In [4], it was shown that for infinite measure-preserving transformations, (spectral) weak mixing is strictly weaker than ergodicity of the Cartesian square.

Another property we consider that is equivalent to weak mixing in the finite measure-preserving case is double ergodicity. This property was introduced by Furstenberg in [9] and was shown to be equivalent to weak mixing for probability-preserving transformations, but was not given a specific name.

A transformation T is said to be **doubly ergodic** if for every pair of sets A and B with positive measure, there exists a positive integer n for which $m(A \cap T^n A)$ and $m(B \cap T^n A)$ are simultaneously nonzero. In the infinite measure-preserving case, double ergodicity is strictly stronger than spectral weak mixing and is properly implied by ergodic Cartesian square [6].

More recently, Aaronson introduced another notion of weak mixing for infinite measure that generalizes rational ergodicity. A transformation T is said to be **rationally weakly mixing** (see [3]) if there exists a measurable set $F \subset X$ of positive finite measure such that for all measurable $A, B \subset F$, we have

$$(3) \quad \frac{1}{a_n(F)} \sum_{k=0}^{n-1} |m(A \cap T^k B) - m(A)m(B)u_k(F)| \rightarrow 0$$

as $n \rightarrow \infty$. Again, it is clear that rational weak mixing reduces to the usual definition of weak mixing in the finite measure-preserving case.

We now describe our main results. In Section 2 we prove that a large class of rank-one transformations are weakly rationally ergodic and discuss the notions of rational ergodicity and bounded rational ergodicity in this context. In Section 3 we construct a class of rank-one transformations that are not rationally weakly mixing; in particular, we obtain a transformation which is rationally ergodic and spectrally weakly mixing but not rationally weakly mixing. This negatively answers a question of Aaronson's. (After this work was completed, we learned that Aaronson had also independently answered this question.) Section 4 shows that rational weak mixing implies double ergodicity and constructs a transformation that is not rationally weakly mixing and which we conjecture to be doubly ergodic. Section 5 proves that the notion of zero-type for infinite measure-preserving transformations (whose spectral definition is equivalent to mixing in the case of probability-preserving transformations) is independent of rational weak mixing. Finally, in Section 6 we present a class of rank-one transformations that are rationally weakly mixing. As remarked in [3], all the examples of rationally weakly mixing transformations constructed in [3] are of the type $T \times S$, where T is an infinite measure-preserving K -automorphism and S is a mildly mixing probability-preserving transformation. These examples have countable Lebesgue spectrum and are of a different nature than our rank-one constructions.

1.1. Acknowledgements. This paper was based on research done by the Ergodic Theory group of the 2012 SMALL Undergraduate Research Project at Williams College. Support for this project was provided by the National Science Foundation REU Grant DMS - 0353634 and the Bronfman Science Center of Williams College. We are indebted to Jon Aaronson for conversations and suggestions during discussions of our work at the 2012 Williams

Ergodic Theory Conference. We would also like to acknowledge the other members of the 2012 Ergodic Theory group: Shelby Heinecke, Emily Wickstrom, and Evangelie Zachos.

1.2. Rank-One Transformations (Basics). We briefly review (rank-one) cutting-and-stacking transformations (see e.g. [13]). A **column** is an ordered collection of pairwise disjoint intervals (called **levels**) in \mathbb{R} , each of the same measure. We think of the levels in a column as being stacked on top of each other, so that the $(j+1)$ -st level is directly above the j -th level. Every column $C = \{J_j\}$ is associated with a natural column map T_C sending each point in J_j to the point directly above it in J_{j+1} . (Note that T_C is undefined on the top level of C .) A **(rank-one) cutting-and-stacking** construction for T consists of a sequence of columns C_n such that:

- (a) The first column C_0 is the unit interval.
- (b) Each column C_{n+1} is obtained from C_n by cutting C_n into $r_n \geq 2$ subcolumns of equal width, adding any number of new levels (called **spacers**) above each subcolumn, and stacking every subcolumn under the subcolumn to its right. In this way, C_{n+1} consists of r_n copies of C_n , possibly separated by spacers.
- (c) The collection of levels $\bigcup_n C_n$ forms a generating semiring for \mathcal{B} .

Observing that $T_{C_{n+1}}$ agrees with T_{C_n} everywhere where T_{C_n} is defined, we then take T to be the limit of T_{C_n} as $n \rightarrow \infty$.

1.3. Rank-One Transformations (Notation). Let T be a rank-one transformation, and fix any column C_n of T . We denote the number of levels in C_n by h_n and write w_n for the width of each level. We denote the height of any level J in C_n by $h(J)$, with the convention that $0 \leq h(J) < h_n$. For each $0 \leq k < r_n$, let $s_{n,k}$ be the number of spacers added above the k -th subcolumn of C_n , and denote the number of levels in the k -th subcolumn (after adding spacers) by $h_{n,k} = h_n + s_{n,k}$.

Define T to be **normal** if $s_{n,r_n-1} > 0$ for infinitely many values of n . (This means that at least one spacer is added above the rightmost subcolumn infinitely many times.) In addition, we say that T has a **bounded number of cuts** if $\sup\{r_n\} < \infty$; this implies that T is partially rigid and of infinite conservative index [5].

Given any level J from C_n and any column C_m of T with $m \geq n$, we define the **descendants** of J in C_m to be the collection of levels in C_m whose disjoint union is J . We denote this set by $D(J, m)$. Occasionally, we will also use $D(J, m)$ to refer to the heights of the descendants of J in C_m .

We say that T **grows exponentially** if $2s_{n,r_n-1} \geq h_{n+1}$ for every n . Intuitively, this means that the upper half of every column C_n consists of spacers

added during the $(n-1)$ -st stage of construction. In particular, the descendants of any level J from an earlier column must lie in the lower half of C_n . Note that any T which grows exponentially is clearly normal.

2. RATIONAL ERGODICITY

In this section, we establish some introductory ideas and prove that a large class of rank-one transformations are rationally ergodic.

We begin with a computational lemma. Suppose that T is a normal rank-one transformation. Then we claim that the partial sums $a_n(J)$ for any level J can be computed from the descendant heights $D(J, N)$ for N sufficiently large. More precisely,

Lemma 2.1. *Let T be a normal rank-one transformation. Fix any level J and $n \in \mathbb{N}$. Then for every N sufficiently large, we have*

$$m(J \cap T^k J) = w_N |D(J, N) \cap (k + D(J, N))|$$

for all $0 \leq k < n$. Consequently,

$$\sum_{k=0}^{n-1} m(J \cap T^k J) = w_N \left(\sum_{k=0}^{n-1} |D(J, N) \cap (k + D(J, N))| \right).$$

Proof. Fix any level J , and let $n \in \mathbb{N}$ be arbitrary. Since T is normal, we can find some column C_N in which all the heights $D(J, N)$ are at most $h_N - n$. For any $0 \leq k < n$ and level $J_i \in D(J, N)$, the image $T^k(J_i)$ is then the level in C_N of height $h(J_i) + k$. The conclusion follows immediately. \square

We will sometimes need to compute $m(J \cap T^k J)$ for $k < 0$. For this, simply observe that

$$m(J \cap T^k J) = m(T^{-k} J \cap J)$$

and

$$|D(J, N) \cap (k + D(J, N))| = |(-k + D(J, N)) \cap D(J, N)|,$$

so in fact Lemma 2.1 holds for all $-n < k < n$.

We thus calculate $D(J, N)$. Suppose that J is a level in C_j of height $h(J)$. Then J splits into r_j levels in C_{j+1} of heights

$$\{h(J)\} \cup \{h(J) + \sum_{k=0}^i h_{j,k} : 0 \leq i < r_j - 1\}.$$

Letting

$$H_j = \{0\} \cup \left\{ \sum_{k=0}^i h_{j,k} : 0 \leq i < r_j - 1 \right\},$$

it follows inductively that

$$D(J, N) = h(J) + H_j \oplus H_{j+1} \oplus \cdots \oplus H_{N-1}.$$

We now show that every normal rank-one transformation satisfies condition (2) for A, B finite unions of levels and F the unit interval. In this context, we note that Aaronson [1, Theorem 6.1] has shown every set of finite measure F contains a dense algebra of sets satisfying (2), but at the same time it is never true that (2) is satisfied for all measurable sets in every set F of finite positive measure [1, Theorem 6.2].

Theorem 2.2. *Let T be a normal rank-one transformation. Then T satisfies condition (2) for A, B finite unions of levels and F the unit interval.*

Proof. Let $F = I$ denote the unit interval. We begin by proving (2) for $A = B = J$, where J is the bottom level of any column C_j . We need to show that

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} m(J \cap T^k J) \rightarrow m(J)^2$$

as $n \rightarrow \infty$. For N sufficiently large (as a function of n), we have

$$\sum_{k=0}^{n-1} m(J \cap T^k J) = w_N \left(\sum_{k=0}^{n-1} |D(J, N) \cap (k + D(J, N))| \right)$$

by Lemma 2.1. Now, writing

$$D(I, N) = H_0 \oplus H_1 \oplus \cdots \oplus H_{N-1}$$

and

$$D(J, N) = H_j \oplus H_{j+1} \oplus \cdots \oplus H_{N-1},$$

we may express $D(I, N) = A \oplus B$ and $D(J, N) = B$ with $A = H_0 \oplus H_1 \oplus \cdots \oplus H_{j-1}$. Noting that $m(J) = 1/|D(I, j)| = 1/|A|$, we thus wish to show

$$\frac{w_N}{a_n(I)} \left(|A|^2 \sum_{k=0}^{n-1} |B \cap (k + B)| \right) \rightarrow 1.$$

We give the term inside the parentheses a combinatorial interpretation. Let $P(n)$ denote the number of ordered quadruplets (a, a', b, b') with $a, a' \in A$ and $b, b' \in B$ for which $0 \leq b - b' < n$. Then the above quotient is precisely $w_N P(n)/a_n(I)$, since $|B \cap (k + B)|$ counts the number of pairs $b, b' \in B$ with $k = b - b'$.

Now let M be the maximum value of $A - A$. We claim that the following inequality holds:

$$\sum_{k=M}^{n-1-M} |(A \oplus B) \cap (k + A \oplus B)| \leq P(n) \leq \sum_{k=-M}^{n-1+M} |(A \oplus B) \cap (k + A \oplus B)|.$$

Indeed, the sum on the left counts the number of quadruplets (a, a', b, b') with $M \leq a - a' + b - b' < n - M$; the sum on the right counts the number of quadruplets with $-M \leq a - a' + b - b' < n + M$. Clearly, any quadruplet

with $M \leq a - a' + b - b' < n - M$ has $0 \leq b - b' < n$. Similarly, any quadruplet with $0 \leq b - b' < n$ has $-M \leq a - a' + b - b' < n + M$. Recalling that $A \oplus B = D(I, N)$, it thus follows that

$$\frac{1}{a_n(I)} \sum_{k=M}^{n-1-M} m(I \cap T^k I) \leq \frac{w_N}{a_n(I)}(P(n)) \leq \frac{1}{a_n(I)} \sum_{k=-M}^{n-1+M} m(I \cap T^k I).$$

Now, M is a fixed constant, independent of n . Furthermore, the sequence $m(I \cap T^k(I))$ is bounded above by 1 but has divergent sum. Hence both sides of the above inequality tend to 1 as $n \rightarrow \infty$, showing that $w_N P(n)/a_n(I) \rightarrow 1$, as desired. This proves (2) for $A = B = J$, where J is the bottom level of any column.

We now prove (2) for J and J' any two levels in the same column. By applying T^{-1} and using the fact that T is measure preserving, we may assume that one of the two levels (say J) is actually the bottom level of the column. Letting $J' = T^d(J)$ for some d , we wish to show

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} m(J \cap T^{k+d} J) \rightarrow m(J)^2.$$

Now, we have from before that

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} m(J \cap T^k J) \rightarrow m(J)^2.$$

Since $m(J \cap T^k J)$ is bounded and $a_n(I) \rightarrow \infty$, the conclusion follows immediately.

Finally, we extend to finite unions of levels. Without loss of generality, we may assume that J and J' are both disjoint unions of images of the same level K . The desired statement then follows from summing together the limits (2) for each pair of images. \square

We now show that under certain conditions, we can extend the results of Theorem 2.2 to all sets A and B (thus proving weak rational ergodicity).

Theorem 2.3. *Let T be an exponentially growing rank-one transformation with a bounded number of cuts. Then T is weakly rationally ergodic.*

Proof. We show that for T satisfying the above hypotheses, it suffices to prove (2) for finite unions of levels (as in Theorem 2.2). Indeed, given arbitrary measurable sets $A, B \subset I$, choose $D \subset I$ a finite union of levels for which $m(D \triangle B) < \varepsilon$. We claim that there is some constant c such that

$$(4) \quad \left| \frac{1}{a_n(I)} \sum_{k=0}^{n-1} m(A \cap T^k B) - \frac{1}{a_n(I)} \sum_{k=0}^{n-1} m(A \cap T^k D) \right| \leq c\varepsilon$$

for every n . Indeed, let $B_0 = B \cap D$, and write $B = B_0 \cup B_1$ and $D = B_0 \cup D_1$. Then the above difference reduces to

$$\left| \frac{1}{a_n(I)} \sum_{k=0}^{n-1} m(A \cap T^k B_1) - \frac{1}{a_n(I)} \sum_{k=0}^{n-1} m(A \cap T^k D_1) \right|.$$

Now, we claim that we can bound

$$(5) \quad \frac{1}{a_n(I)} \sum_{k=0}^{n-1} m(A \cap T^k B_1) \leq cm(B_1)$$

for some c . Applying this bound with D_1 in place of B_1 will bound (4) by $c(m(B_1) + m(D_1)) \leq 2c\varepsilon$, as desired.

For each $m \in \mathbb{N}$, define $M_m = \max(D(I, m))$. (That is, let M_m be the height of the uppermost descendant of I in C_m .) Clearly, $\{M_m\}$ is an increasing sequence. For any fixed n , if we choose m such that $M_{m-1} \leq n-1 < M_m$, we have

$$\sum_{k=0}^{n-1} m(A \cap T^k B_1) \leq \sum_{k=0}^{n-1} m(I \cap T^k B_1) \leq \sum_{k=0}^{M_m} m(I \cap T^k B_1)$$

and

$$\sum_{k=0}^{M_{m-1}} m(I \cap T^k I) \leq \sum_{k=0}^{n-1} m(I \cap T^k I) = a_n(I).$$

To prove (5), it thus suffices to find some c such that

$$(6) \quad \sum_{k=0}^{M_m} m(I \cap T^k B_1) \leq cm(B_1) \sum_{k=0}^{M_{m-1}} m(I \cap T^k I)$$

for every m .

Now observe that the sets $T^k I$ with $-M_m \leq k \leq M_m$ cover each point of I exactly $|D(I, m)|$ times. Indeed, consider the column C_m and fix any $x \in I$. Let x be contained in J , where J is some level from $D(I, m)$. For any level J' in $D(I, m)$, we claim that there is exactly one value of k between $-M_m$ and M_m for which $T^k J' \cap J \neq \emptyset$. Indeed, suppose $0 \leq k \leq M_m$ and $T^k J' \cap J \neq \emptyset$. Any forward image $T^k J'$ with $0 \leq k \leq M_m$ is just a translation upwards by k levels, since $h_m \geq 2M_m$. (This is implied by our hypothesis that T is exponentially growing.) Hence in this case k must equal $h(J) - h(J')$. On the other hand, suppose $-M_m \leq k < 0$ and $T^k J' \cap J \neq \emptyset$. Then $J' \cap T^{-k} J \neq \emptyset$, and exactly the same argument shows that $-k = h(J') - h(J)$ (i.e., $k = h(J) - h(J')$). The claim is then immediate.

We thus have

$$\sum_{k=-M_m}^{M_m} m(I \cap T^k B_1) = \sum_{k=-M_m}^{M_m} m(T^k I \cap B_1) = |D(I, m)| m(B_1)$$

and

$$\sum_{k=-M_{m-1}}^{M_{m-1}} m(I \cap T^k I) = |D(I, m-1)|.$$

Hence

$$\sum_{k=-M_m}^{M_m} m(I \cap T^k B_1) = \left(\frac{|D(I, m)|}{|D(I, m-1)|} \right) m(B_1) \left(\sum_{k=-M_{m-1}}^{M_{m-1}} m(I \cap T^k I) \right)$$

and so

$$\sum_{k=0}^{M_m} m(I \cap T^k B_1) \leq \left(\frac{|D(I, m)|}{|D(I, m-1)|} \right) m(B_1) \left(2 \left(\sum_{k=0}^{M_{m-1}} m(I \cap T^k I) \right) - 1 \right).$$

But $|D(I, m)|/|D(I, m-1)| = r_{m-1}$, and T has a bounded number of cuts. We thus easily obtain (6). Hence (4) holds, and we can approximate B with D a finite union of levels. Applying a similar argument to A shows that it suffices to prove (2) for all A, B finite unions of levels, which is the content of Theorem 2.2. \square

We now consider some alternate notions of rational ergodicity, also due to Aaronson [1]. For any measurable function f , recall the notation

$$S_n(f) = \sum_{k=0}^{n-1} f \circ T^k.$$

We say that T is **rationally ergodic** if there exists a set F of positive finite measure which satisfies a **Renyi inequality**; i.e., there is some constant M such that

$$(7) \quad \int_F (S_n(1_F))^2 dm \leq M \left(\int_F S_n(1_F) dm \right)^2$$

for every $n \in \mathbb{N}$. If this inequality holds only on a subset $\{n_i\} \subset \mathbb{N}$, we say that T is **subsequence rationally ergodic**. Some authors adopt this as the definition of rational ergodicity instead (see e.g. [8]). It was shown in [1] that rational ergodicity implies weak rational ergodicity. It is not currently known whether these notions are equivalent.

We say that T is **boundedly rationally ergodic** (see [2]) if there exists a set F of positive finite measure such that

$$\sup_{n \geq 1} \left\| \frac{1}{a_n(F)} S_n(1_F) \right\|_\infty < \infty.$$

In [2], it was shown that bounded rational ergodicity is a strictly stronger property than rational ergodicity. It is not difficult to see that the proof of Theorem 2.3 (in particular, the establishment of (5)) yields bounded rational ergodicity for the transformations in question. Aaronson proved in [2] that every dyadic tower over the adding machine is boundedly rationally ergodic; Theorem 2.3 extends this result to a larger class of transformations and uses a different approach. Some interesting examples of exponentially growing rank-one transformations with a bounded number of cuts include:

- (a) Hajian-Kakutani skyscraper-type constructions [12]:

$$r_n = 2, \{s_{n,0} = 0, s_{n,1} \geq 2h_n\}.$$

(When $s_{n,1} = 2h_n + 1$ the transformation is spectral weakly mixing, see [5]).

- (b) Chacón-like constructions:

$$r_n = 3, \{s_{n,0} = 0, s_{n,1} = 1, s_{n,2} \geq 3h_n + 1\}.$$

(When $s_{n,2} = 3h_n + 1$ the transformation has infinite ergodic index, see [5], but is not power weakly mixing, see [10].

We now prove a slightly different version of Theorem 2.3 without the hypothesis of a bounded number of cuts. Let T be the Hajian-Kakutani skyscraper transformation. We claim that T is subsequence rationally ergodic on $F = I = (0, 1)$. We proceed by proving a Renyi inequality for I with $M = 2$ and $n_i = 2^i$. Note that $S_n(1_I)(x)$ is equal to the number of k with $0 \leq k \leq n - 1$ for which $T^k(x) \in I$.

Theorem 2.4. *The Hajian-Kakutani skyscraper transformation is subsequence rationally ergodic on I .*

Proof. We verify the Renyi inequality (7) above for $n = 2^m$. Consider the m -th iteration of the construction of T , corresponding to the column C_m with 4^m levels (each of length 2^{-m}). Let $D(I, m) = \{I_j\}$ be the set of descendants of I in C_m , and order $\{I_j\}$ by height of appearance in C_m so that I_1 is the lowermost level of $\{I_j\}$ in C_m and I_{2^m} is the uppermost.

Let the heights of the levels $\{I_j\}$ in C_m be denoted by $\{h(I_j)\}$. It is immediate that $S_n(1_I)$ is constant on each level I_j , and that the value of $S_n(1_I)$ on a fixed level I_l is equal to the cardinality of the intersection $(h(I_l) + \{0, 1, \dots, n - 1\}) \cap \{h(I_j)\}$. Now, all the descendants of I lie in the lower half of C_m , so the maximum difference between any two elements of $\{h(I_j)\}$ is less than $2^m = n$. It follows that (in fact) $h(I_j) \in (h(I_l) + \{0, 1, \dots, n - 1\})$ for every $j \geq l$. Restricting the domain of $S_n(1_I)$ to I , we may thus write

$$S_n(1_I) = \sum_{j=1}^n (n + 1 - j) 1_{I_j}$$

where $n + 1 - j$ is the value of S_n on I_j . Proving (6) is then equivalent to showing

$$\sum_{j=1}^n \frac{(n+1-j)^2}{n} \leq 2 \left(\sum_{j=1}^n \frac{n+1-j}{n} \right)^2.$$

Multiplying through by n^2 and reindexing yields

$$n \left(\sum_{j=1}^n j^2 \right) \leq 2 \left(\sum_{j=1}^n j \right)^2$$

and the result then follows from the formulas for power sums. \square

We leave it to the reader to extend the proof of Theorem 2.4 to other rank-one transformations. In particular, the above proof immediately generalizes to show that any exponentially growing T is rationally ergodic along the subsequence $\{M_m + 1\}$.

3. RATIONAL WEAK MIXING

In this section, we present a large class of transformations that are *not* rationally weakly mixing. We obtain as a corollary the existence of transformations which are rationally ergodic and spectrally weakly mixing, but not rationally weakly mixing.

We begin with an example of a rank-one transformation which is subsequence rationally weakly mixing.

First, consider the Chacón rank-one transformation T constructed by starting with the unit interval, cutting each column in half, and adding a single spacer on top of the right subcolumn at every step [7]. This transformation is finite measure-preserving and weakly mixing; thus, it is rationally weakly mixing. We claim that (in particular) T is rationally weakly mixing on the unit interval $I = (0, 1)$.

It is clear from the definition of weak rational ergodicity that if T is weakly rationally ergodic on F , then T is weakly rationally ergodic on any subset of F . Moreover, it was shown in [3] that for T rationally weakly mixing, the class of sets F satisfying (2) is the same as the class of sets F satisfying (3). This establishes the claim.

Now let

$$\phi_n(A, B) = \frac{1}{a_n(I)} \sum_{k=0}^{n-1} |m(A \cap T^k B) - m(A)m(B)u_k(I)|$$

be the quotient from (3), and let D_m denote the collection of dyadic intervals of the form $(i/2^m, (i+1)/2^m)$ for $0 \leq i < 2^m$. Since D_1 is a finite collection and T is rationally weakly mixing, there exists some natural number m_1 such that for all $A, B \in D_1$ we have $\phi_{m_1}(A, B) < 1/2$. We claim that in fact this inequality is true for every rank-one transformation \tilde{T} which shares its first m_1 stages of construction with T (i.e., $\tilde{C}_n = C_n$ for all $n < m_1$). Indeed, for $A, B \subset I$, the value of $\phi_{m_1}(A, B)$ depends only on the first m_1 stages of the construction of T , since the heights $D(I, m_1)$ are all less than $h_{m_1} - m_1$.

We now define our desired transformation. We begin by following the construction of the transformation T as described above, until we reach C_{m_1} . Then, at the m_1 -th iteration, we add $2h_{m_1}$ spacers above the right subcolumn (doubling the height of C_{m_1}). Now, adding one spacer at each subsequent iteration gives another finite measure-preserving transformation, which is also weakly mixing. Hence, there is some $m_2 > m_1$ such that $\phi_{m_2}(A, B) < 1/4$ for all $A, B \in D_1 \cup D_2$.

We thus continue adding a single spacer at each step until we reach C_{m_2} , at which point we add $2h_{m_2}$ spacers (again doubling the size of our column). Proceeding inductively in this manner, we obtain a cutting-and-stacking transformation T and a sequence (m_i) such that for each i , $\phi_{m_i}(A, B) < 1/2^i$ for all $A, B \in D_1 \cup D_2 \cup \dots \cup D_i$. The result is an invertible, infinite measure-preserving transformation which is rationally weakly mixing along (m_i) for dyadic intervals.

In order to extend to all subsets of I , we use the following result due to Aaronson [3]:

Lemma 3.1. *Let T be an invertible measure-preserving transformation on a Polish space X , and assume that T is rationally ergodic on some open set F . Suppose there is a countable base \mathcal{C} for the topology of F such that for every finite subcollection $\{C_i\} \subset \mathcal{C}$, there exists a finite subcollection $\{D_i\} \subset \mathcal{C}$ which is disjoint and has the same union. Then to establish rational weak mixing, it suffices to prove condition (2) for elements of \mathcal{C} .*

Lemma 3.1 also holds for establishing subsequence rational weak mixing, so long as rational ergodicity is known along the same subsequence. Since the transformation T above may be expressed as a dyadic tower over the adding machine, T is rationally ergodic [2]. It follows from Lemma 3.1 that T is subsequence rationally weakly mixing.

We now present a large class of examples that are not rationally weakly mixing. It will be convenient to write

$$u_k(A, B) = \frac{m(A \cap T^k B)}{m(A)m(B)}$$

so that for A, B of positive measure, we can divide (2) by $m(A)m(B)$ to obtain

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |u_k(A, B) - u_k(F)| \rightarrow 0.$$

It is not difficult to see that in this case we must have $a_n(F)/a_n(A, B) \rightarrow 1$ [3]. This yields the following theorem:

Theorem 3.2. *Let T be a rank-one transformation constructed by cutting C_n in half and adding at least $c_n \geq 2h_n$ spacers on top of the right subcolumn at every step. Then T is not rationally weakly mixing.*

Proof. We prove by contradiction. Suppose that T is rationally weakly mixing on some set F . Choose a level J which is at least $(3/4)$ -full of F , and let J_1 and J_2 be the left and right halves of J . By applying T^{-1} to F , we may assume that J is the bottom level of some column C_j . Now, both J_1 and J_2 intersect F in positive measure, so

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |u_k(J_1 \cap F) - u_k(F)| \rightarrow 0$$

and

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |u_k(J_1 \cap F, J_2 \cap F) - u_k(F)| \rightarrow 0.$$

Moreover, $a_n(F)/a_n(J_1 \cap F) \rightarrow 1$. Multiplying through by this limit and using the triangle inequality, we obtain

$$(8) \quad \frac{1}{a_n(J_1 \cap F)} \sum_{k=0}^{n-1} |u_k(J_1 \cap F, J_2 \cap F) - u_k(J_1 \cap F)| \rightarrow 0.$$

Now, fix k and suppose that $u_k(J_1 \cap F) > 0$. Then $m(J_1 \cap T^k J_1) > 0$, so for sufficiently large N we have $k \in D(J_1, N) - D(J_1, N)$. Similarly, if $u_k(J_1 \cap F, J_2 \cap F) > 0$, then $m(J_1 \cap T^{k+h_j} J_1) = m(J_1 \cap T^k J_2) > 0$, which implies that $k + h_j \in D(J_1, N) - D(J_1, N)$. Hence we cannot have both $u_k(J_1 \cap F)$ and $u_k(J_1 \cap F, J_2 \cap F)$ nonzero, since then we would have $h_j \in (D(J_1, N) - D(J_1, N)) - (D(J_1, N) - D(J_1, N))$. As $D(J_1, N) = \{0, h_{j+1}\} \oplus \{0, h_{j+2}\} \oplus \cdots \oplus \{0, h_{N-1}\}$, this is easily seen to be impossible (given the fact that $c_n \geq 2h_n$ for all n).

It is then immediate that

$$|u_k(J_1 \cap F, J_2 \cap F) - u_k(J_1 \cap F)| \geq u_k(J_1 \cap F)$$

for every k . Indeed, if $u_k(J_1 \cap F) = 0$ then we are done; otherwise, $u_k(J_1 \cap F, J_2 \cap F)$ is 0. It follows that the quotient (8) is bounded below by 1, which is a contradiction. This shows that T is not rationally weakly mixing. \square

In particular, we obtain the following:

Corollary 3.3. *There exist transformations which are rationally ergodic and spectrally weakly mixing but not rationally weakly mixing.*

Proof. Consider any rank-one transformation T constructed by cutting C_n in half and adding $2h_n + 1$ spacers on top of the right subcolumn at every step. This is rationally ergodic by Theorem 2.3, and is spectrally weakly mixing by a standard argument [5]. By Theorem 3.2, however, T is not rationally weakly mixing. \square

This negatively answers a question of Aaronson's.

We now extend Theorem 3.2 to other rank-one transformations. Define

$$H = \bigcup_{j=0}^{\infty} H_j \setminus \{0\}$$

and observe that the elements of H are increasing when listed in the obvious order. (Begin with successive elements of H_0 , followed by successive elements of H_1 , and so on.) We say that a rank-one transformation is **steep** if $s_{i+1} \geq 4s_i$ for every pair of successive $s_i, s_{i+1} \in H$. Clearly, the transformations of Theorem 3.2 are steep. In general, such transformations can be constructed by adding an exponentially increasing number of spacers above successive subcolumns. It is not difficult to see that steep transformations satisfy a nice algebraic uniqueness property; namely, every integer k has at most one representation

$$(9) \quad k = \sum_{s \in H} e_s s$$

with $e_s \in \{-2, -1, 0, 1, 2\}$. The proof of Theorem 3.2 immediately generalizes to show that steep transformations are not rationally weakly mixing.

Theorem 3.4. *Let T be a steep rank-one transformation. Then T is not rationally weakly mixing.*

Proof. We sketch the proof and leave the details to the reader. As before, we proceed by contradiction. Suppose T is rationally weakly mixing on F , and let J be a level $(3/4)$ -full of F . Without loss of generality, we may assume that J is the bottom level of some column C_j . Now, there must exist at least two descendants J_1 and J_2 of J in C_{j+1} that have positive intersection with F . For these levels, we have $J_2 = T^d J_1$ for some $d \in H_j - H_j$. It then suffices to show that d cannot be contained in $(D(J_1, N) - D(J_1, N)) - (D(J_1, N) - D(J_1, N))$, which follows from the uniqueness of (9). \square

4. RELATION TO DOUBLE ERGODICITY

In this section we show that rational weak mixing implies double ergodicity and present an example suggesting the converse implication is false.

We begin by proving that rational weak mixing on F implies double ergodicity for subsets of F .

Theorem 4.1. *Suppose that T is rationally weakly mixing on F . Then T is doubly ergodic for all $A, B \subset F$.*

Proof. Let $A, B \subset F$, and fix $\delta > 0$ such that

$$\delta < \frac{1}{2} \min(m(A)^2, m(A)m(B)).$$

Since T is rationally weakly mixing on F ,

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |m(A \cap T^k A) - m(A)^2 u_k(F)| \rightarrow 0$$

and

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |m(A \cap T^k B) - m(A)m(B)u_k(F)| \rightarrow 0.$$

Summing these together, we obtain (by contradiction) that there exists a positive integer k for which $u_k(F) > 0$ and

$$|m(A \cap T^k A) - m(A)^2 u_k(F)| + |m(A \cap T^k B) - m(A)m(B)u_k(F)| < \delta u_k(F).$$

We thus have

$$|m(A \cap T^k A) - m(A)^2 u_k(F)| < \delta u_k(F)$$

and so

$$m(A \cap T^k A) > u_k(F)(m(A)^2 - \delta) > 0$$

for this k . Similarly,

$$m(A \cap T^k B) > u_k(F)(m(A)m(B) - \delta) > 0.$$

By construction of δ , this shows that T is doubly ergodic on F . \square

We now extend this result to all of X . It was shown in [1] that if T is weakly rationally ergodic on F , it is weakly rationally ergodic on any finite union $F_N = F \cup T(F) \cup \dots \cup T^{N-1}(F)$. It follows that the analogous statement holds for rational weak mixing, giving the following theorem:

Theorem 4.2. *Suppose that T is rationally weakly mixing. Then T is doubly ergodic.*

Proof. Let T be rationally weakly mixing on F , and suppose that T is not doubly ergodic. Fix $A, B \subset X$ for which the double ergodicity condition fails; i.e., choose A and B such that for every n , either $m(A \cap T^n A) = 0$ or $m(A \cap T^n B) = 0$. Since F sweeps out X , there is some N for which F_N intersects both A and B in positive measure. Then $\tilde{A} = F_N \cap A$ and $\tilde{B} = F_N \cap B$ are sets of positive measure which fail the double ergodicity condition. But T is doubly ergodic on F_N , a contradiction. \square

It is worth noting that (in general) the class of sets on which T is doubly ergodic is *not* a hereditary ring. For example, let T be any doubly ergodic transformation on X , and define S on $X \times \{0, 1\}$ by $S(x, 0) = (T(x), 1)$ and $S(x, 1) = (x, 0)$. Then S is doubly ergodic on both $X \times \{0\}$ and $X \times \{1\}$, but not doubly ergodic on all of $X \times \{0, 1\}$. (Let $A = X \times \{0\}$ and $B = X \times \{1\}$.)

We now investigate whether rational weak mixing is strictly stronger than double ergodicity. It will be useful for us consider transformations that are “almost” steep. Recall that T is steep if for any pair of successive elements s_i, s_{i+1} in $H = (H_0 \cup H_1 \cup \dots) \setminus \{0\}$, we have $s_{i+1} \geq 4s_i$. Now, suppose T is constructed so that:

- (a) Each column C_n is cut into at least three subcolumns ($r_n \geq 3$).
- (b) We add zero spacers above the first subcolumn and one spacer above the second ($s_{n,0} = 0$ and $s_{n,1} = 1$).
- (c) We add a sufficient number of spacers above each subsequent subcolumn so that

$$\sum_{k=0}^i h_{n,k} \geq 4 \left(\sum_{k=0}^{i-1} h_{n,k} \right)$$

for every $2 \leq i \leq r_n - 1$.

Then T is “almost” steep, in the sense that $s_{i+1} < 4s_i$ only when s_i and s_{i+1} are the first two nonzero elements of some H_n . For such T , we can still extract a (slightly technical) algebraic uniqueness condition in the spirit of (9). Indeed, let

$$B_n = \{h_{n,0}, h_{n,0} + h_{n,1}\} \times \{h_{n,0}, h_{n,0} + h_{n,1}\}$$

and define

$$A_n = (H_n \times H_n) \setminus (\Delta H_n \cup B_n).$$

(Here, $\Delta H_n = \{(x, x) : x \in H_n\}$.) Then for any $a, b, a', b' \in A_n$ and $-M_n \leq k, k' \leq M_n$, the equality

$$(10) \quad k + a - b = k' + a' - b'$$

implies

$$a = a', b = b', k = k'.$$

(The proof of this is not difficult and is left to the reader.) Before we proceed, it will be useful to establish following lemma:

Lemma 4.3. *Let J be any level, and fix N sufficiently large. Suppose $(a, b) \in A_N$ and $-M_N \leq k \leq M_N$. Then*

$$m(J_1 \cap T^{k+a-b}J) = \frac{1}{r_N} m(J_1 \cap T^k J).$$

Proof. By Lemma 2.1, we have

$$m(J_1 \cap T^k J) = w_N |D(J, N) \cap (k + D(J, N))|$$

and

$$m(J_1 \cap T^{k+a-b}J) = w_{N+1}|D(J, N+1) \cap (k+a-b+D(J, N+1))|.$$

By uniqueness of (10), every representation of $k+a-b$ as an element of $D(J, N+1) - D(J, N+1)$ corresponds to exactly one representation of k as an element of $D(J, N) - D(J, N)$, and vice-versa. Hence

$$m(J_1 \cap T^{k+a-b}J) = \frac{w_{N+1}}{w_N}m(J_1 \cap T^k J) = \frac{1}{r_N}m(J_1 \cap T^k J),$$

as desired. \square

We now show that if T is almost steep and $\{r_n\}$ is sufficiently large, T cannot be rationally weakly mixing.

Theorem 4.4. *Let T be a rank-one transformation. Suppose that T is almost steep (as described above), and that*

$$\sum_{n=0}^{\infty} \frac{1}{r_n} < \infty.$$

Then T is not rationally weakly mixing.

Proof. We begin by proving that T is not rationally weakly mixing on levels. Let J be the bottom level of any column C_j , and let J_1 and J_2 be any two descendants of J in C_{j+1} . Then $J_1 = T^d J_2$ for some $d \in H_j - H_j$. As in Theorem 3.2, it suffices to disprove the convergence

$$(11) \quad \frac{1}{a_n(J_1)} \sum_{k=0}^{n-1} |u_k(J_1) - u_k(J_1, J_2)| \rightarrow 0.$$

To do this, define

$$P_m = \sum_{k=-M_m}^{M_m} |m(J_1 \cap T^k J_1) - m(J_1 \cap T^{k+d} J_1)|$$

and

$$Q_m = \sum_{k=-M_m}^{M_m} m(J_1 \cap T^k J_1).$$

For m sufficiently large, $R_m = P_m/Q_m$ approximates the quotient (11), so it is enough to show that R_m is bounded below by some positive constant.

Any choice of $(a, b) \in A_m$ and $-M_m \leq k \leq M_m$ yields a unique number $k+a-b$ between $-M_{m+1}$ and M_{m+1} . Hence

$$P_{m+1} = \sum_{k=-M_{m+1}}^{M_{m+1}} |m(J_1 \cap T^k J_1) - m(J_1 \cap T^{k+d} J_1)|$$

$$\begin{aligned}
&\geq \sum_{(a,b) \in A_m} \sum_{k=-M_m}^{M_m} |m(J_1 \cap T^{k+a-b} J_1) - m(J_1 \cap T^{k+a-b+d} J_1)| \\
&= \frac{1}{r_m} \left(\sum_{(a,b) \in A_m} \sum_{k=-M_m}^{M_m} |m(J_1 \cap T^k J_1) - m(J_1 \cap T^{k+d} J_1)| \right) \\
&= \frac{|A_m|}{r_m} P_m.
\end{aligned}$$

Moreover, the same argument as in Theorem 2.3 shows

$$Q_m = \sum_{k=-M_m}^{M_m} m(J_1 \cap T^k J_1) = |D(J_1, m)| m(J_1)$$

from which it follows that

$$Q_{m+1} = r_m Q_m.$$

We thus obtain

$$R_{m+1} \geq \frac{|A_m|}{r_m^2} R_m.$$

Now, $|A_m| = r_m^2 - r_m - 2$, so R_m is bounded below by

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{r_k} - \frac{2}{r_k^2} \right) R_0$$

which is a positive constant by the hypotheses of the theorem. This bounds (10) from below along the sequence $\{M_m + 1\}$. Since T is rationally ergodic along the same sequence by Theorem 2.4, it follows that T is not rationally weakly mixing. \square

We now show that T is doubly ergodic for levels, suggesting that rational weak mixing is strictly stronger than double ergodicity.

Lemma 4.5. *The transformation T above is doubly ergodic for levels.*

Proof. We check that for any pair of levels A and B , there exists an integer n such that both $m(A \cap T^n A) > 0$ and $m(B \cap T^n A) > 0$. Without loss of generality, we may assume that A is the bottom level of some column C_j and that $B = T^d A$. It then suffices to prove there exists an n such that both n and $n + d$ are in $D(A, N) - D(A, N)$ (for N sufficiently large). This is easy; simply choose

$$n = h_{j+1,0} + \cdots + h_{j+d,0}.$$

Then

$$n + d = ((2h_{j+1,0} + 1) + \cdots + (2h_{j+d,0} + 1)) - (h_{j+1,0} + \cdots + h_{j+d,0}),$$

as desired. \square

5. INDEPENDENCE FROM ZERO-TYPE

We now show that (subsequence) rational weak mixing and zero-type are independent (i.e., do not imply each other). We say that T is **zero-type** if $m(A \cap T^n A) \rightarrow 0$ for all sets A of finite measure [11]. It is well-known that in order to show a conservative ergodic transformation is zero-type, it suffices to check this convergence for a single set A of positive finite measure [11]. We show that every steep transformation with an increasing number of cuts is zero-type.

Theorem 5.1. *Let T be a steep rank-one transformation, and suppose that $\{r_n\}$ is nondecreasing with $\sup\{r_n\} = \infty$. Then T is zero-type.*

Proof. Consider $I = (0, 1)$. For N sufficiently large, we have

$$m(I \cap T^k I) = \frac{|D(I, N) \cap (k + D(I, N))|}{|D(I, N)|}.$$

Now, $|D(I, N) \cap (k + D(I, N))|$ counts the number of representations

$$(12) \quad k = \sum_{i=0}^{N-1} (d_i - d'_i)$$

with $d_i, d'_i \in H_i$. (Recall that $D(I, N) = H_0 \oplus H_1 \oplus \cdots \oplus H_{N-1}$.) If $k \notin D(I, N) - D(I, N)$, then $m(I \cap T^k I) = 0$, so suppose that $k \in D(I, N) - D(I, N)$. Then there is at least one representation

$$(13) \quad k = \sum_{i=0}^{N-1} (x_i - x'_i)$$

with $x_i, x'_i \in H_i$. If we fix n and suppose that $x_n - x'_n \neq 0$, the uniqueness of (9) implies any other representation (12) of k must have $d_i = x_i$ and $d'_i = x'_i$. In particular, the only indices i at which (12) can differ from (13) are those for which $x_i - x'_i = 0$. In these cases, we must choose $d_i = d'_i$, but otherwise there are no restrictions (i.e., $d_i = d'_i$ can be any element of H_i). Hence

$$|D(I, N) \cap (k + D(I, N))| = \prod_{x_i - x'_i = 0} |H_i|$$

with the product being taken over all i for which $x_i - x'_i = 0$. Since

$$|D(I, N)| = \prod_{i=0}^{N-1} |H_i|$$

it follows that

$$m(I \cap T^k I) = \left(\prod_{x_i - x'_i \neq 0} |H_i| \right)^{-1}.$$

Now, if $k > M_n$, then the representation (13) of k must have $x_m - x'_m \neq 0$ for some $m \geq n$. This implies that

$$m(I \cap T^k I) \leq \frac{1}{|H_m|} = \frac{1}{r_m} \leq \frac{1}{r_n},$$

which shows $m(I \cap T^k I) \rightarrow 0$ as $k \rightarrow \infty$. Hence T is zero-type, as desired. \square

We thus have:

Theorem 5.2. *There exist transformations that are zero-type but not rationally weakly mixing.*

As a partial converse, observe that the subsequence rationally weakly mixing transformation of Section 3 is not zero-type. (Indeed, $m(I \cap T^{h_i} I) \geq 1/2$ for every i .) Thus,

Theorem 5.3. *There exist transformations that are subsequence rationally weakly mixing but not zero-type.*

6. EXAMPLES OF RATIONAL WEAK MIXING

We end with an example of an infinite measure-preserving rank-one transformation that is rationally weakly mixing. Let T be a Chacón-like transformation ($r_n = 3$, $\{s_{n,0} = 0, s_{n,1} = 1, s_{n,2} \geq 3h_n + 1\}$) with enough spacers added above every third subcolumn so as to have $h_{n+1} = 3^c h_n$ for some fixed natural number $c \geq 2$. Then $h_n = 3^{cn}$ and

$$D(I, n) = H_0 \oplus H_1 \oplus \cdots \oplus H_{n-1}$$

where $H_i = \{0, h_i, 2h_i + 1\}$.

Theorem 6.1. *The transformation T above is rationally weakly mixing.*

Proof. We begin by proving rational weak mixing for levels. Let $J = J_1$ be the bottom level of C_j , and let $J_2 = T^d J_1$. Then (as in the proof of Theorem 4.4), it suffices to show the convergence (11). Now, for any n , we may choose m such that $M_{m-1} \leq n - 1 < M_m$. Then the quotient (11) is asymptotically bounded above by $P_m/Q_{m-1} = 3P_m/Q_m$, so it suffices to prove $P_m/Q_m \rightarrow 0$ as $m \rightarrow \infty$.

Observe that by the triangle inequality,

$$\begin{aligned} P_m &= \sum_{k=-M_m}^{M_m} |m(J \cap T^k J) - m(J \cap T^{k+d} J)| \\ &\leq \sum_{j=0}^{d-1} \sum_{k=-M_m}^{M_m} |m(J \cap T^{k+j} J) - m(J \cap T^{k+1+j} J)|. \end{aligned}$$

Since each of the d outer sums on the right differs from the $j = 0$ sum by a finite number of terms, it suffices to show that

$$\frac{\sum_{k=-M_m}^{M_m} |m(J \cap T^k J) - m(J \cap T^{k+1} J)|}{\sum_{k=-M_m}^{M_m} m(J \cap T^k J)} \rightarrow 0.$$

Fix m and define a function on $D(J, m) \times D(J, m)$ as follows. Every element $d - d' \in D(J, m) - D(J, m)$ has a natural representation

$$(14) \quad d - d' = \left(\sum_{i=j}^{m-1} d_i \right) - \left(\sum_{i=j}^{n-1} d'_i \right)$$

with $d_i, d'_i \in \{0, h_i, 2h_i + 1\}$. Replacing each instance of $2h_i + 1$ with $2h_i$ in (14) yields a sum of the form

$$(15) \quad \sum_{i=j}^{m-1} \varepsilon_i 3^{ci}$$

with $\varepsilon_i \in \{-2, -1, 0, 1, 2\}$. This defines a function g mapping every element $(d, d') \in D(J, m) \times D(J, m)$ to a vector $\varepsilon = \{\varepsilon_i\}_{i=j}^{m-1}$ in $\{-2, -1, 0, 1, 2\}^{m-j}$ via (14).

For each fixed $\varepsilon \in \{-2, -1, 0, 1, 2\}^{m-j}$, define a function $\tilde{\varepsilon}$ on $D(J, m) - D(J, m)$ by

$$\tilde{\varepsilon}(k) = |g^{-1}(\varepsilon) \cap \{(d, d') | d - d' = k, d, d' \in D(J, m)\}|.$$

That is, $\tilde{\varepsilon}(k)$ counts the number of pairs (d, d') in $g^{-1}(\varepsilon)$ with $d - d' = k$. Then we claim that the following properties hold:

Lemma 6.2. *Fix $\varepsilon \in \{-2, -1, 0, 1, 2\}^{m-j}$. We have:*

(a) *For any k ,*

$$\sum_{\varepsilon} \tilde{\varepsilon}(k) = |D(J, m) \cap (k + D(J, m))|.$$

(b) *For $p \in \{-2, -1, 0, 1, 2\}$, let a_p be the number of ε_i in (15) equal to p . Then*

$$\sum_k \tilde{\varepsilon}(k) = 3^{a_0} 2^{a_1 + a_{-1}}.$$

Proof. For (a), simply observe that both the left and right-hand expressions count the number of pairs (d, d') for which $d - d' = k$.

For (b), observe that the sum on the left counts the number of pairs (d, d')

whose associated vector is ε . Now, if $\varepsilon_i = 0$ in (15), then we must have $d_i = d'_i$ in (14), and there are three ways that this can happen. Similarly, if $\varepsilon_i = 1$, then either $d_i = h_i$ and $d'_i = 0$, or $d_i = 2h_i + 1$ and $d'_i = h_i$. Proceeding in this manner, a counting argument yields the desired equality. \square

(*Proof of Theorem 12, continued.*) Using the results of Lemma 6.1, we thus need to show

$$(16) \quad \frac{\sum_{k=-M_m}^{M_m} \left| \sum_{\varepsilon} \tilde{\varepsilon}(k) - \sum_{\varepsilon} \tilde{\varepsilon}(k+1) \right|}{\sum_{k=-M_m}^{M_m} \sum_{\varepsilon} \tilde{\varepsilon}(k)} \rightarrow 0.$$

By the triangle inequality, it suffices to prove this convergence after exchanging the order of summation in both the numerator and denominator. To this end, we claim there exists a nonincreasing function $c(t)$ which converges to 0 such that

$$R(\varepsilon) := \frac{\sum_k |\tilde{\varepsilon}(k) - \tilde{\varepsilon}(k+1)|}{\sum_k \tilde{\varepsilon}(k)} \leq c(a_1 + a_{-1})$$

for each ε . Once we have such a function, we obtain the following bound for large enough L :

$$\begin{aligned} \sum_{\varepsilon} \sum_{k=-M_m}^{M_m} |\tilde{\varepsilon}(k) - \tilde{\varepsilon}(k+1)| &\leq \sum_{\varepsilon} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k) c(a_1 + a_{-1}) \\ &\leq c(L) \sum_{\substack{\varepsilon \text{ with} \\ a_1 + a_{-1} \geq L}} \left(\sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k) \right) \\ &\quad + c(0) \sum_{\substack{\varepsilon \text{ with} \\ a_1 + a_{-1} < L}} \left(\sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k) \right). \end{aligned}$$

Dividing this by the denominator of (16) yields

$$\frac{\sum_{\varepsilon} \sum_{k=-M_m}^{M_m} |\tilde{\varepsilon}(k) - \tilde{\varepsilon}(k+1)|}{\sum_{\varepsilon} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k)} \leq c(L) + c(0)d(L)$$

where

$$d(L) = \frac{\sum_{\substack{\varepsilon \text{ with} \\ a_1 + a_{-1} < L}} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k)}{\sum_{\varepsilon} \sum_{k=-M_m}^{M_m} \tilde{\varepsilon}(k)}.$$

Now, we claim that $d(L) \rightarrow 0$ as $m \rightarrow \infty$. For simplicity, suppose $j = 0$. Then $\sum_{p=-2}^2 a_p = m$ for every vector ε , so we can view the summation over ε as a sum over all possible 5-tuples $(a_0, a_{-1}, a_1, a_{-2}, a_2)$ of natural numbers with $\sum_{p=-2}^2 a_p = m$. By Lemma 5.2, we thus have

$$\begin{aligned} d(m) &= \frac{\sum_{\substack{\varepsilon \text{ with} \\ a_1 + a_{-1} < m}} 3^{a_0} 2^{a_1 + a_{-1}}}{\sum_{\varepsilon} 3^{a_0} 2^{a_1 + a_{-1}}} \\ &\leq \frac{2^m \sum_{\varepsilon} \binom{n}{a_0, a_{-1}, a_1, a_{-2}, a_2} 3^{a_0}}{\sum_{\varepsilon} \binom{n}{a_0, a_{-1}, a_1, a_{-2}, a_2} 3^{a_0} 2^{a_1} 2^{a_{-1}}} \\ &= 2^m \frac{7^n}{9^n}. \end{aligned}$$

Next, we will show that $s(n) := \sup \{R(\varepsilon) : a_1 + a_{-1} = n\}$ converges to 0. This will conclude the proof, because $c(n) := \sup \{s(m) : m \geq n\}$ will be a nonincreasing function converging to 0.

Fix ε , and let a be the minimum element of $D(J, m) - D(J, m)$ for which $\tilde{\varepsilon}(a) > 0$. Any $k \in D(J, m) - D(J, m)$ is expressible as $k = \sum \varepsilon_i 3^{ci} + \sum (+1) + \sum (-1)$, with the $+1$'s and -1 's coming from choosing $2h_i + 1$ for d_i and d'_i in (14). We now ask: how many $+1$'s and -1 's do we have for $k = a$? We have only one way of obtaining $\varepsilon_i = 2$ in (14): namely, $(2h_i + 1) - 0$. Similarly, we only have one way of obtaining $\varepsilon_i = -2$: namely, $0 - (2h_i + 1)$. This introduces a_2 number of $+1$'s and a_{-2} number of -1 's. We have three ways of obtaining $\varepsilon_i = 0$, none of which introduce a net number of $+1$'s or -1 's. For $\varepsilon_i = 1$, we have two possibilities: either $h_i - 0$ or $(2h_i + 1) - h_i$. Since we want to minimize a , we choose the former. Similarly, for $\varepsilon_i = -1$, we must have either $0 - h_i$ or $h_i - (2h_i + 1)$, and to minimize a we choose the latter. It thus follows that a has a_2 number of $+1$'s and $a_{-2} + a_{-1}$ number of -1 's; moreover, $\tilde{\varepsilon}(a) = 3^{a_0}$. It is then not difficult to see that

$$\tilde{\varepsilon}(a + k) = 3^{a_0} \binom{a_1 + a_{-1}}{k}$$

for all $0 \leq k \leq a_1 + a_{-1}$, and is 0 otherwise.

Letting $n = a_1 + a_{-1}$, we thus have

$$R(\varepsilon) = \frac{1}{2^n} \left(\sum_{k=0}^{n-1} \left| \binom{n}{k} - \binom{n}{k+1} \right| + 2 \right).$$

Suppose $n = 2l - 1$. (The case when n is even is dealt with similarly.) Since

$$\left| \binom{n}{k} - \binom{n}{k-1} \right| = \binom{n+1}{k} \left| \frac{(n+1) - 2k}{n+1} \right|,$$

the above expression yields

$$\begin{aligned} R(\varepsilon) &= \frac{1}{2^n} \left(\sum_{k=1}^n \binom{n+1}{k} \left| \frac{(n+1) - 2k}{n+1} \right| + 2 \right) \\ &= \frac{1}{2^n} \left(\sum_{k=1}^{2l-1} \binom{2l}{k} \left| \frac{l-k}{l} \right| + 2 \right) \\ &\leq \frac{1}{2^n} \left(2 \sum_{k=0}^l \binom{2l}{k} \left(\frac{l-k}{l} \right) \right). \end{aligned}$$

Using the combinatorial identity

$$\sum_{k=0}^l \binom{2l}{k} \left(\frac{l-k}{l} \right) = \frac{l+1}{2l} \binom{2l}{l+1},$$

we obtain

$$R(\varepsilon) \leq \frac{1}{2^{2l}} \binom{2l}{l+1}.$$

It is not difficult to see that this goes to 0 as a function of l , thus proving that T is rationally weakly mixing for levels. By Theorem 2.3 and Lemma 3.1, it follows that T is rationally weakly mixing. \square

REFERENCES

- [1] J. Aaronson. Rational ergodicity and a metric invariant for Markov shifts. *Israel J. Math.*, 27(2):93–123, 1977.
- [2] J. Aaronson. Rational ergodicity, bounded rational ergodicity and some continuous measures on the circle. *Israel J. Math.*, 33(3-4):181–197 (1980), 1979. A collection of invited papers on ergodic theory.
- [3] J. Aaronson. Rational weak mixing in infinite measure spaces. <http://arxiv.org/abs/1105.3541>, 2012.
- [4] Jonathan Aaronson, Michael Lin, and Benjamin Weiss. Mixing properties of Markov operators and ergodic transformations, and ergodicity of Cartesian products. *Israel J. Math.*, 33(3-4):198–224 (1980), 1979. A collection of invited papers on ergodic theory.
- [5] T. Adams, N. Friedman, and C. E. Silva. Rank-one weak mixing for nonsingular transformations. *Israel J. Math.*, 102:269–281, 1997.
- [6] A. Bowles, L. Fidkowski, A. E. Marinello, and C. E. Silva. Double ergodicity of non-singular transformations and infinite measure-preserving staircase transformations. *Illinois J. Math.*, 45(3):999–1019, 2001.
- [7] R. V. Chacon. Weakly mixing transformations which are not strongly mixing. *Proc. Amer. Math. Soc.*, 22:559–562, 1969.

- [8] P. R. Cirilo, Y. Lima, and E. Pujals. Law of large numbers for certain cylinder flows. <http://arxiv.org/abs/1108.3519>, 2011.
- [9] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory*. Princeton University Press, Princeton, 1981.
- [10] Kate Gruher, Fred Hines, Deepam Patel, Cesar E. Silva, and Robert Waelde. Power weak mixing does not imply multiple recurrence in infinite measure and other counterexamples. *New York J. Math.*, 9:1–22 (electronic), 2003.
- [11] A. B. Hajian and S. Kakutani. Weakly wandering sets and invariant measures. *Trans. Amer. Math. Soc.*, 110:136–151, 1964.
- [12] A. B. Hajian and S. Kakutani. Example of an ergodic measure preserving transformation on an infinite measure space. In *Contributions to Ergodic Theory and Probability (Proc. Conf., Ohio State Univ., Columbus, Ohio, 1970)*, pages 45–52. Springer, Berlin, 1970.
- [13] C. E. Silva. *Invitation to ergodic theory*, volume 42 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2008.

(Irving Dai) HARVARD COLLEGE, UNIVERSITY HALL, CAMBRIDGE, MA 02138, USA
E-mail address: ifdai@college.harvard.edu

(Xavier Garcia) UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455-0213, USA
E-mail address: garci363@umn.edu

(Tudor Pădurariu) UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90095-1555, US
E-mail address: tudor-pad@yahoo.com

(Cesar Silva) DEPARTMENT OF MATHEMATICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267, USA
E-mail address: csilva@williams.edu